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Calculating multiloop integrals using dimensional recurrence relation and $\mathcal{D}$-analyticity

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We review the method of the calculation of multiloop integrals recently suggested in Ref.[1]. A simple method of derivation of the dimensional recurrence relation suitable for automatization is given. Some new analytic results are given.

1. Introduction

Recently, in Ref [1] a method of multiloop integrals evaluation based on $\mathcal{D}$ recurrence relations [2] and $\mathcal{D}$-analyticity was suggested (DRA method). In this contribution we give a brief review of this method. We also provide a simple method of derivation of the dimensional recurrence relation well-suited for automatization.

2. DRA method

The DRA method has been described in detail in Ref. [1]. It consists of the following steps:

1. Make sure all master integrals in subtopologies are known. If it is not so, start from calculating them.

2. Pass to a suitable master integral $J^{(\mathcal{D})}$. It is convenient to choose a master integral which is finite in the basic stripe. For this purpose, e.g., increase powers of some massive propagators.

3. Construct the dimensional recurrence relation for this master integral. The general form of this recurrence is

$$J^{(\mathcal{D}-2)} = C(\mathcal{D})J^{(\mathcal{D})} + R(\mathcal{D}), \tag{1}$$

where $C(\mathcal{D})$ is some rational function and $R(\mathcal{D})$ is a non-homogeneous part constructed of the master integrals of subtopologies in $\mathcal{D}$ dimensions.

4. Find a general solution of this recurrence relation

$$J^{(\mathcal{D})} = \Sigma^{-1}(\mathcal{D})\omega(z) + J^{(\mathcal{D})}_{\text{ih}}, \tag{2}$$

where $J^{(\mathcal{D})}_{\text{ih}}$ is a specific solution of the inhomogeneous equation, $\Sigma^{-1}(\mathcal{D})$ is the solution of the homogeneous equation, and $\omega(z) = \omega(\exp(i\pi\mathcal{D}))$ is arbitrary periodic function.

5. Fix the singularities of $\omega(z)$ by analysing the analytical properties of the master integrals and summing factor $\Sigma(\mathcal{D})$.

6. If needed, fix the remaining constants from the value of the integral at some space-time dimension $\mathcal{D}$.

Step 5 is the key point of the DRA approach. In order to perform this step one needs to determine the position and order of the poles of $\Sigma(\mathcal{D})J^{(\mathcal{D})}$ on suitable vertical stripe of width 2 in the complex plane of $\mathcal{D}$ (*basic stripe*). This information can be extracted from the parametric representation of the integral, either manually or (semi-)automatically using the `FIESTA` code [3]. In general, the number and the order of poles essentially depend on the choice of the master integral $J^{(\mathcal{D})}$, the summing factor $\Sigma(\mathcal{D})$, and the basic stripe. The proper choice may essentially simplify the last step of the approach.

Before we proceed to the example of the application of the DRA method, we would like to derive formulas convenient for the automatic derivation of the dimensional recurrence relation.

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3. Dimensional recurrence relation

The original derivation of the dimensional recurrence relation [2] is based on the parametric representation. For the integral without numerator which can be represented by some graph the final formula has the form of some sum over the graph trees. For the automatic calculation it may be desirable to have the possibility to obtain the dimensional recurrence relation without any reference to the graph and/or to the parametric representation. In this Section we obtain the corresponding formulas using the Baikov's approach which consists of the "changing of integration variables" from loop momenta to scalar products (or denominators) [4]. We briefly review the derivation of the corresponding transformation keeping also $\mathcal{D}$-dependent factors omitted in the original derivation of Ref. [4].

Assume that we are interested in the calculation of the L-loop integral depending on E linearly independent external momenta $p_1,\ldots,p_E$. There are $N = L(L+1)/2 + LE$ scalar products depending on the loop momenta l_i:

$$s_{ij} = s_{ji} = l_i \cdot q_j\,; \quad i = 1,\ldots,L; \quad j = 1,\ldots,K, \tag{3}$$

where $q_{1,\ldots,L} = l_{1,\ldots,L}$, $q_{L+1,\ldots,K} = p_{1,\ldots,E}$, and $K = L + E$.

The loop integral has the form

$$\begin{aligned} J^{(\mathcal{D})}(\mathbf{n}) &= \int \frac{d^{\mathcal{D}}l_L \ldots d^{\mathcal{D}}l_1}{\pi^{L\mathcal{D}/2}} j(n_1,\ldots,n_N) \\ &= \int \frac{d^{\mathcal{D}}l_L \ldots d^{\mathcal{D}}l_1}{\pi^{L\mathcal{D}/2} D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} \end{aligned} \tag{4}$$

where the scalar functions D_α are linear polynomials with respect to s_{ij}. The functions D_α are assumed to be linearly independent and to form a complete basis in the sense that any non-zero linear combination of them depends on the loop momenta, and any s_{ik} can be expressed in terms of D_α.

The integral $J^{(\mathcal{D})}(\mathbf{n})$ can be considered as a function of N integer variables. It is convenient [1] to introduce the operators A_i, B_i which act on such functions as

$$\begin{aligned} (A_i f)(\ldots,n_i,\ldots) &= n_i f(\ldots,n_i+1,\ldots) \\ (B_i f)(\ldots,n_i,\ldots) &= f(\ldots,n_i-1,\ldots) \end{aligned} \tag{5}$$

Let us first transform the innermost integral $\int d^{\mathcal{D}}l_1/\pi^{\mathcal{D}/2}$ in Eq. (4). The integrand j depends on l_1 via the scalar products s_{1i} $(i = 1\ldots L+E)$. Writing $l_1 = l_{1\parallel} + l_{1\perp}$, where $l_{1\parallel}$ is the projection of l_1 on the hyperplane spanned by $q_2 \ldots q_K$, we obtain

$$\begin{aligned} \frac{d^{\mathcal{D}}l_1}{\pi^{\mathcal{D}/2}} &= \frac{d^{\mathcal{D}-K+1}l_{1\perp}}{\pi^{(\mathcal{D}-K+1)/2}} \frac{d^{K-1}l_{1\parallel}}{\pi^{(K-1)/2}} \\ &= \frac{\left(\mu \frac{V(q_1,\ldots q_K)}{V(q_2,\ldots q_K)}\right)^{(\mathcal{D}-K-1)/2} ds_{11}}{\Gamma[(\mathcal{D}-K+1)/2]} \\ &\times \frac{ds_{12}\ldots ds_{1K}}{\pi^{(K-1)/2}\sqrt{\mu^{K-1}V(q_2,\ldots q_K)}} \end{aligned} \tag{6}$$

where $V(q_1,\ldots q_K) = \det\{s_{ij}|_{i,j=1\ldots K}\}$ is a Gram determinant constructed on the vectors $q_1,\ldots q_K$ and $\mu = \pm 1$ for the Euclidean/pseudoEuclidean case, respectively. Note that

$$V(q_1,\ldots,q_K) = P(D_1,\ldots,D_N)$$

is a K-degree polynomial of D_α.

Repeating the same transformation for $l_2,\ldots,l_L$, we finally obtain

$$\begin{aligned} J(\mathbf{n}) &= \frac{\pi^{(L-N)/2}\mu^{L\mathcal{D}/2-N}}{\Gamma\left[(\mathcal{D}-K+1)/2,\ldots,(\mathcal{D}-E)/2\right]} \\ &\times \int \left(\prod_{i=1}^{L}\prod_{j=i}^{K} ds_{ij}\right) \frac{[V(q_1,\ldots,q_K)]^{(\mathcal{D}-K-1)/2}}{[V(p_1,\ldots,p_E)]^{(\mathcal{D}-E-1)/2}} j(\mathbf{n}) \end{aligned} \tag{7}$$

In order to use this formula in explicit calculations, we also need to determine the limits of integration over the s_{ij} variables. However, for algebraic manipulations we only need to keep in mind that the integration by part does not generate any surface terms.

The lowering dimensional recurrence relation is immediately obtained by replacing $\mathcal{D} \to \mathcal{D}+2$ in Eq. (7) and comparing the resulting expression with the original one [1]. We obtain

$$J^{(\mathcal{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L \left[V(p_1,\ldots,p_E)\right]^{-1}}{(\mathcal{D}-E-L+1)_L} \times \left(P(B_1,\ldots,B_N)\, J^{(\mathcal{D})}\right)(\mathbf{n})\,. \quad (8)$$

In order to obtain the relation between master integrals, we have to use IBP reduction for the right-hand side of Eq. (8). The complexity of this reduction strongly depends on the integrals appearing in the right-hand side. The lowering dimensional recurrence relation (8) contains integrals with indices shifted by at most $K = L + E$ in comparison with the integral in the left-hand aside.

The raising dimensional recurrence relation is more "economic" from this point of view. In the original Tarasov's derivation the parametric representation of the loop integral was used. For the integral given by some graph, the result is expressed in terms of the trees of this graph. However, for the automatic derivation of the raising recurrence relation this formula may be inconvenient. Therefore, it is desirable to be able to obtain the raising recurrence relation without any reference to the graph. In order to obtain the raising recurrence relation we use the identity

$$\det\left\{2^{\delta_{ij}}\frac{\partial}{\partial s_{ij}}|_{i,j=1\ldots L}\right\}\left[V(q_1,\ldots,q_K)\right]^{\alpha} = (2\alpha)_L V(p_1,\ldots,p_E)\left[V(q_1,\ldots,q_K)\right]^{\alpha-1} \quad (9)$$

The proof of this identity is based on the Carl Jacobi theorem about determinants and will be presented elsewhere. The raising dimensional recurrence relation is obtained by replacing $\mathcal{D} \to \mathcal{D} - 2$ in Eq. (7), substituting $V(p_1,\ldots,p_E)\left[V(q_1,\ldots,q_K)\right]^{(\mathcal{D}-K-3)/2}$ with the derivative and integrating by part. We obtain

$$\begin{aligned} J^{(\mathcal{D}-2)}(\mathbf{n}) &= (-\mu/2)^L \\ &\times \int \frac{d^{\mathcal{D}}l_L \ldots d^{\mathcal{D}}l_1}{\pi^{L\mathcal{D}/2}} \det\left\{\frac{2^{\delta_{ij}}\partial}{\partial s_{ij}}|_{i,j=1\ldots L}\right\} j(\mathbf{n}) \\ &= (\mu/2)^L \\ &\times \left(\det\left\{2^{\delta_{ij}}\frac{\partial D_k}{\partial s_{ij}}A_k|_{i,j=1\ldots L}\right\} J^{(\mathcal{D})}\right)(\mathbf{n})\,. \quad (10)\end{aligned}$$

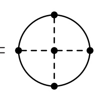
Comparing Eq. (10) with Tarasov's formula we obtain for the case of integral corresponding to some graph:

$$\det\left\{2^{\delta_{ij}-1}\frac{\partial D_k}{\partial s_{ij}}A_k|_{i,j=1\ldots L}\right\} = \sum_{\text{trees}} A_{i_1}\ldots A_{i_L}\,,$$

where the sum goes over all trees of the graph, and $i_1,\ldots,i_L$ enumerate the chords of the tree.

4. Example

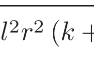
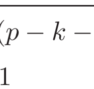
Let us demonstrate the application of the method on the calculation of the following four-loop vacuum integral:

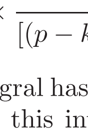
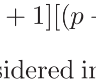
$J^{(\mathcal{D})} =$ 

$$= \int \frac{d^{\mathcal{D}}k\, d^{\mathcal{D}}l\, d^{\mathcal{D}}r\, d^{\mathcal{D}}p}{\pi^{2\mathcal{D}}k^2 l^2 r^2 (k+l+r)^2 \left[(p-k-l)^2+1\right]} \times \frac{1}{[(p-k)^2+1][p^2+1][(p+r)^2+1]}$$

This integral has been considered in Refs. [5,6]. In Ref. [5] this integral has been evaluated numerically using the Laporta's defference equation method. In Ref. [6] this integral has been considered using the dimensional recurrence relation. However, in that paper in order to fix the periodic function parametrizing the homogeneous solution we had to resort to the Laporta's difference equation. Here we present the derivation entirely based on the DRA method. This derivation serves solely as the illustration of the DRA method. The final result for arbitrary $\mathcal{D}$ coincides with the result of Ref. [6].

1. There are four master integrals in the subtopologies:

$J_1^{(\mathcal{D})} \equiv$ , $J_2^{(\mathcal{D})} =$ ,

$J_3^{(\mathcal{D})} =$ , $J_4^{(\mathcal{D})} =$ . (11)

These integrals are expressed in terms of Γ-functions, see, e.g., Ref. [6].

2. The integral has no ultraviolet divergence for $\mathcal{D} < 4$. At $\mathcal{D} = 4$ the integral has a simple pole. The integral has a simple pole also at $\mathcal{D} = 2\frac{2}{3}$ due to the infrared infrared divergence. This divergence comes from the region where k, l, and r are small. Subtracting from the integrand the quantity $\left[k^2 l^2 r^2 (k+l+r)^2 \left(p^2+1\right)^4\right]^{-1}$, corresponding to a scaleless integral, we easily establish, that $J^{(\mathcal{D})}$ is finite when $2 < \mathrm{Re}\mathcal{D} < 2\frac{2}{3}$. We choose the basic stripe as $S = \{\mathcal{D}| \quad \mathrm{Re}\,\mathcal{D} \in (2, 4]\}$.

3. The dimensional recurrence for $J^{(\mathcal{D})}$ reads

$$J^{(\mathcal{D}+2)} = -\frac{48(3\mathcal{D}-11)(3\mathcal{D}-7)}{(\mathcal{D}-3)_4(\mathcal{D}-2)^2} J^{(\mathcal{D})} + c_1^{(\mathcal{D})} J_1^{(\mathcal{D})} + c_2^{(\mathcal{D})} J_2^{(\mathcal{D})} + c_3^{(\mathcal{D})} J_3^{(\mathcal{D})} + c_4^{(\mathcal{D})} J_4^{(\mathcal{D})},$$

where $c_i^{(\mathcal{D})}$ are some rational functions not presented here for brevity (see Ref. [6]).

4. The summing factor obeys the equation

$$\frac{\Sigma(\mathcal{D})}{\Sigma(\mathcal{D}+2)} = -\frac{48(3\mathcal{D}-11)(3\mathcal{D}-7)}{(\mathcal{D}-3)_4(\mathcal{D}-2)^2} \quad (12)$$

Since $J^{(\mathcal{D})}$ has simple poles in the basic stripe at $\mathcal{D} = 2\frac{2}{3}, 4$, we choose the summing factor to have zeros at these points. Namely, we choose

$$\Sigma(\mathcal{D}) = \frac{\cos\left(\frac{\pi\mathcal{D}}{2} + \frac{\pi}{6}\right)\Gamma^2(\mathcal{D}-3)\Gamma\left(\frac{\mathcal{D}}{2}\right)}{8^{\mathcal{D}}\Gamma(3-\mathcal{D})\Gamma\left(\frac{3\mathcal{D}}{2} - \frac{11}{2}\right)} \quad (13)$$

The general solution of the dimensional recurrence has the form

$$\Sigma(\mathcal{D})\, J^{(\mathcal{D})} = \omega(z) - \sum_{i=1}^{4} s_i(\mathcal{D}) \quad (14)$$

$$s_i(\mathcal{D}) = \sum_{k=0}^{\infty} t_i(\mathcal{D}+2k) \quad (15)$$

$$t_i(\mathcal{D}) = \Sigma(\mathcal{D})\, c_i^{(\mathcal{D})} J_i^{(\mathcal{D})} \quad (16)$$

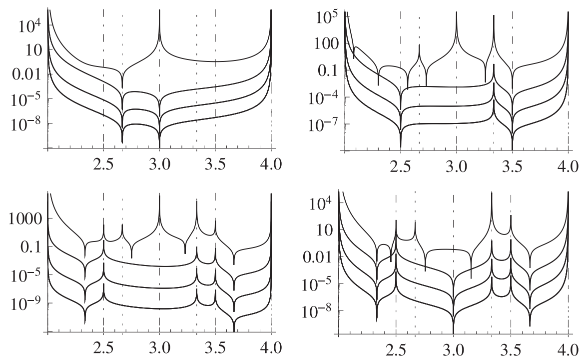


Figure 1. Pole structure of functions t_1, t_2 (upper row), t_3, t_4 (lower row). We plot $|t_i(\mathcal{D}+2k)|$ for $k = 0, \ldots, 3$.

5. The left-hand side of Eq. (15) has no singularities on S, so the right-hand side should also be a holomorphic function. The functions $t_i(\mathcal{D}+2k)$ have poles at $\mathcal{D} = \mathcal{D}_{1-6}$ where

$$\mathcal{D}_1 = 2\frac{1}{2}, \mathcal{D}_2 = 2\frac{2}{3}, \mathcal{D}_3 = 3,$$
$$\mathcal{D}_4 = 3\frac{1}{3}, \mathcal{D}_5 = 3\frac{1}{2}, \mathcal{D}_6 = 4,$$

The pole structure of $t_i(\mathcal{D}+2k)$ is demonstrated in Fig. 1. Note that $s_i(\mathcal{D})$ may have poles only in the points where the individual terms $t_i(\mathcal{D}+2k)$ are singular.

Therefore, we obtain

$$\omega(z) = \sum_{l=1}^{6} \sum_{r=1}^{r_l} c_l^r \left[\cot\left(\frac{\pi}{2}(\mathcal{D}-\mathcal{D}_l)\right)\right]^r + \mathrm{const}, \quad (17)$$

where r_l is the order of the pole at $\mathcal{D} = \mathcal{D}_l$, and the coefficients c_l^r should be chosen so as to cancel all singularities in the right-hand side of Eq. (15). Their determination is reduced to the solution of some linear system which we do not present here for brevity.

As it is shown in Ref. [1], an L-loop integral is bounded in the limit $\mathcal{D} \to \pm i\infty$ by $z^{\pm L/4} |\log z|^{\nu}$, where ν is some irrelevant exponent. Using this fact and the explicit form of $\Sigma(\mathcal{D})$, it is easy to establish that $\Sigma(\mathcal{D}) J^{(\mathcal{D})}$ and $s_i(\mathcal{D})$ fall down when $\mathcal{D} \to \pm i\infty$. Therefore, the constant in Eq. (17) should be chosen in such a way that $\omega(z)$ falls down when $z \to 0, \infty$. The first term in Eq. (17) has different limits when $\mathcal{D} \to \pm i\infty$, and we obtain

$$\text{const} = -\sum_{l=1}^{6} \sum_{r=1}^{r_l} c_l^r (-i)^r = -\sum_{l=1}^{6} \sum_{r=1}^{r_l} c_l^r (+i)^r \tag{18}$$

Eqs. (15),(17), and (18) entirely determine $J^{(\mathcal{D})}$ for arbitrary $\mathcal{D}$. However, we may want to find the coefficients c_l^r in Eq. (17) explicitely. Using the fast convergence of sums in $s_i(\mathcal{D})$ and keeping in mind the possibility to use the `pslq` algorithm [7], we find that values of all these coefficients are compatible with zero, at least, up to 10^{-500}. Therefore, we conclude that

$$\omega(z) \overset{500}{=} 0, \tag{19}$$

where $\overset{500}{=}$ denotes the equality checked numerically with 500 digits.

6. Our consideration allowed us to fix all constants within the method. Adopting the guess (19), we obtain

$$J^{(\mathcal{D})} = -\Sigma^{-1}(\mathcal{D}) \sum_{i=1}^{4} s_i(\mathcal{D})$$

This result coincides with that of Ref. [6]. Using the `pslq` algorithm, we can express the expansion near $\mathcal{D} = 4$ in terms of conventional ζ-values:

$$\begin{aligned}
J^{(4-2\epsilon)} &\overset{300}{=} \frac{\epsilon^3 \Gamma[-1+\epsilon]^4}{1+\epsilon} \left[1 + 2\epsilon^3 + 3\epsilon^4 + O(\epsilon^5)\right] \\
&\times \left[5\zeta_5 - \left(7\zeta_3^2 + \frac{11\pi^6}{378}\right)\epsilon + \left(\frac{\pi^4\zeta_3}{30} + 212\zeta_7\right)\epsilon^2 \right. \\
&- \left(\frac{29213\pi^8}{32400} - 1820\zeta_{2,6} - 5038\zeta_3\zeta_5\right)\epsilon^3 \\
&+ \left(\frac{13255\zeta_9}{3} + \frac{731\pi^4\zeta_5}{6} - \frac{2006\pi^6\zeta_3}{189} + \frac{1006\zeta_3^3}{3}\right)\epsilon^4 \\
&\left. + O(\epsilon^5)\right]
\end{aligned}$$

Note that the factor in the first line is chosen so as to provide the uniform transcendentality weight in the rest of expansion. In Ref. [5] the first term of the above expansion has been found analytically and the remaining terms have been found numerically with 40-digit precision.

Let us also present the result for the expansion around $\mathcal{D} = 3$, which can be important for the calculations in hot QCD:

$$\begin{aligned}
J^{(3-2\epsilon)} &\overset{300}{=} \frac{\Gamma[-1/2+\epsilon]^4}{1+\epsilon} \left[\frac{\pi^2}{96} + \frac{11\zeta_3}{16}\epsilon \right. \\
&\left. + \left(\frac{271\pi^4}{2880} + \pi^2 \log 2 - \frac{5\zeta_3}{2} - \frac{41\pi^2}{48}\right)\epsilon^2 + O(\epsilon^3)\right]
\end{aligned}$$

In Ref. [6] the first term of the above expansion has been found analytically and the second term has been obtained numerically.

5. Conclusion

We have briefly reviewed the method of calculation of multiloop integrals based on the $\mathcal{D}$-recurrence and $\mathcal{D}$-analyticity. The method appeares to be powerful enough to deal with the most complicated cases. We have also derived convenient formulas, Eqs. (8) and (10), suitable for the automatic derivation of the Tarasov's dimensional recurrence. For a specific four-loop master integral we have presented in analytic form several terms of the expansion around $\mathcal{D} = 4$ and $\mathcal{D} = 3$.

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